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The deformation retract of the complex projective space and its topological folding

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The deformation retract of the complex projective space into itself and also after the isometric and topological folding was studied and discussed. Theorems concerning these deformation retracts are deduced.

1. Introduction

It is known that the real projective n-space, p^n, as the quotient space of the n-sphere, s^n, obtained by identifying antipodal points, p^n, is a compact connected n-dimensional manifold. The complex projective n-space cp^n, is a compact connected $2n$-dimensional manifold: in complex $(n+1)$-space, c^{n+1}, consider the subspace defined by $|z| = 1$ where if $z = (z_0, z_1, \ldots, z_n)$, we define $|z|^2 = |z_0|^2 + |z_1|^2 + \cdots + |z_n|^2)$; this space is just s^{2n+1}. We identify z, z' on s^{2n+1} if $z' = cz$, where c is a complex number of absolute value 1; the resulting quotient space is called cp^n, the fibres of the map f: $s^{2n+1} \to cp^n$ are circles [1–4].

A map $F: M \to N$, where M, N are C^∞ Riemannian manifolds of dimensions m, n, respectively, is said to be an isometric folding of M into N, if and only if for any piecewise geodesic path γ: $J \to M$, the induced path $F \circ \gamma$: $J \to N$ is a piecewise geodesic and of the same length as γ, $J = [0,1]$. If F does not preserve length, then F is a topological folding [5, 6]. By using the Lagrangian equations

$$\frac{d}{ds}\left(\frac{\partial T}{\partial \psi_i'}\right) - \frac{\partial T}{\partial \psi_i} = 0 \qquad i = 1, 2, \ldots, 2n+1. \quad (1)$$

we determined a geodesic $s^{2n} \subset s^{2n+1}$ [7] which is the deformation retract of $\{s^{2n+1} - p_i\}$. Also by the above equation we obtain $s^{2n-1} \subset s^{2n}$ which is geodesic and it is the deformation retract of $\{s^{2n} - p_i\}$, p_i is any antipodal points. Consequently, $cp^{n-1} \subset cp^n$, cp^{n-1} is the deformation retract of $\{cp^n - p_i\}$, $p_i \in cp^n$. In this paper we discuss the relation between the deformation retract of $\{cp^n - p_i\}$ and the deformation retract of the isometric and topological folding of $\{cp^n - p_i\}$ [8–11].

A subset A of a topological space M is a deformation retract of M if there exists a retraction R: $M \to A$ and a homotopy f: $M \times I \to M$ such that [12]

$$\left.\begin{aligned} f(x,0) &= x \qquad (2a)\\ f(x,1) &= R(x) \qquad (2b)\end{aligned}\right\} \quad x \in M$$

$$f(a,t) = a,\ a \in A \quad \text{and} \quad t \in I = [0,1] \qquad (2c)$$

2. The main results

Let s^{2n+1} be the sphere $x_1^2 + x_2^2 + \cdots + x_{2n+2}^2 = 1$; the parametric equation of this sphere is

$$r = \left(\cos\psi_1 \prod_{k=1}^{2n} \sin\psi_{k+1}, \prod_{k=1}^{2n+1} \sin\psi_k, \ldots, \cos\psi_{i-1} \prod_{k=i}^{2n+1} \right.$$
$$\left. \times \sin\psi_k, \ldots, \cos\psi_{2n+1}\right) \quad i = 3, 4, 5, \ldots, 2n+1 \quad (3)$$

For any point $\alpha \in cp^n$, then $\alpha = \{(z; z'): z, z' \in s^{2n+1}$ such that $z = cz'$, $|c| = 1\}$. If we take $c = e^{i\pi}$ hence, $(z_0, z_1, \ldots, z_n) \in cp^n$ or $(x_1, x_2, \ldots, x_{2n+2}; -x_1, -x_2, \ldots, -x_{2n+2}) \in cp^n$. Consider the parametric from

$$\alpha = \left[\left(\cos\psi_1 \prod_{k=1}^{2n} \sin\psi_{k+1}, \prod_{k=1}^{2n+1} \sin\psi_k, \ldots, \cos\psi_{i-1}\right.\right.$$
$$\times \prod_{k=i}^{2n+1} \sin\psi_k, \ldots, \cos\psi_{i-1} \prod_{k=i}^{2n+1} \sin\psi_k, \ldots,$$
$$\left.\cos\psi_{2n+1}\right); e^{i\pi}\left(\cos\psi_1 \prod_{k=1}^{2n} \sin\psi_{k+1},\right.$$
$$\prod_{k=1}^{2n+1} \sin\psi_k, \ldots, \cos\psi_{i-1} \prod_{k=i}^{2n+1} \sin\psi_k, \ldots,$$
$$\left.\left.\cos\psi_{2n+1}\right)\right] =: [\beta; -\beta]. \quad (4)$$

From the equations $ds^2 = (\sum_{j=1}^{2n+2} dx_j^2; -\sum_{j=1}^{2n+2} dx_j^2)$ and $T = \frac{1}{2} ds^2$, we have

$$T = \frac{1}{2}\left[\left(\prod_{i=2}^{2n+1} \sin^2\psi_i \psi_1'^2 + \prod_{i=3}^{2n+1} \sin^2\psi_i \psi_2'^2\right.\right.$$
$$+ \prod_{i=4}^{2n+1} \sin^2\psi_i \psi_3'^2 + \prod_{i=5}^{2n+1} \sin^2\psi_i \psi_4'^2 + \prod_{i=6}^{2n+1}$$
$$\left.\sin^2\psi_i \psi_5'^2 + \cdots + \sin^2\psi_{2n+1} \psi_{2n}'^2 + \psi_{2n+1}'^2\right);$$
$$-\left(\prod_{i=2}^{2n+1} \sin^2\psi_i \psi_1'^2 + \prod_{i=3}^{2n+1} \sin^2\psi_i \psi_2'^2 + \prod_{i=4}^{2n+1}\right.$$

$$\times \sin^2\psi_i\psi_3'^2 + \prod_{i=5}^{2n+1} \sin^2\psi_i\psi_4'^2 + \prod_{i=6}^{2n+1} \sin^2\psi_1\psi_5'^2$$

$$+ \cdots + \sin^2\psi_{2n+1}\psi_{2n}'^2 + \psi_{2n+1}'^2\Big)\Big] \quad (5)$$

The Lagrangian equations for cp^n are

$$\frac{d}{ds}\left(\frac{\partial T}{\partial \psi_e'}\right) - \frac{\partial T}{\partial \psi_e} = 0 \qquad e = 1,2,3,\ldots,2n+1 \quad (6)$$

$$\frac{d}{ds}\left(\prod_{i=2}^{2n+1} \sin^2\psi_i\psi_1'\right) = 0 \quad (7)$$

$$\frac{d}{ds}\left(\prod_{i=3}^{2n+1} sin^2\psi_i\psi_2'\right)$$

$$- \sin\psi_2\cos\psi_2 \prod_{i=3}^{2n+1} \sin^2\psi_i\psi_1'^2 = 0 \quad (8)$$

$$\frac{d}{ds}\left(\prod_{i=4}^{2n+1} \sin^2\psi_i\psi_3'\right) - \left(\sin\psi_3\cos\psi_3 \prod_{i=4}^{2n+1} \sin^2\psi_i\psi_2'^2\right.$$

$$\left.+ \sin\psi_3\cos\psi_3\sin^2\psi_2 \prod_{i=4}^{2n+1} \sin^2\psi_i\psi_1'^2\right) = 0 \quad (9)$$

$$\vdots$$

$$\frac{d}{ds}\left(\psi_{2n+1}'^2\right) - \left[\sin\psi_{2n+1}\cos\psi_{2n+1}\left(\psi_{2n}'^2\right.\right.$$

$$+ \sin^2\psi_{2n}\psi_{2n-1}'^2 + \sum_{l=2}^{2n-1}\prod_{k=1}^{l}$$

$$\left.\left.\times \sin^2\psi_{2n+1-k}\psi_{2n+1-(l+1)}'^2\right)\right] \cdots (2n+1)$$

From Equation 7 then

$$\frac{d}{ds}\left(\prod_{i=2}^{2n+1} \sin^2\psi_i\psi_1'\right) = 0, \text{ or } \prod_{i=2}^{2n+1} \sin^2\psi_i\psi_1'$$

$$= \text{constant}, \alpha_1\text{: say.}$$

If $\alpha_1 = 0$, then $\psi_1' = 0$ [1] or $\psi_2 = 0$ or π, take $\psi_1' \neq 0$ and $\psi_2 = 0$.

$(x_1, x_2, x_3, \ldots, x_{2n+2};\ -x_2, \ldots,\ -x_{2n+2})$ in the form

$$\begin{aligned}
x_1 &= 0\\
x_2 &= 0\\
x_3 &= \cos\psi_2 \prod_{k=3}^{2n+1} \sin\psi_k\\
&\vdots\\
x_i &= \cos\psi_{i-1} \prod_{k=i}^{2n+1} \sin\psi_k,\ i = 4,\ldots,2n+1\\
&\vdots\\
x_{2n+2} &= \cos\psi_{2n+1}\\
-x_1 &= 0\\
-x_2 &= 0\\
&\vdots\\
-x_{2n+2} &= -\cos\psi_{2n+1}
\end{aligned} \quad (10)$$

From the system of Equation 10 we obtain $(0, 0, x_3, \ldots, x_{2n+2};\ -0,\ -0,\ -x_3, \ldots,\ -x_{2n+2})$ which is $cp^{n-1} \subset cp^n$; this represents a geodesic. The deformation retract of cp^n may be defined by

$$\Phi\colon (cp^n - \{p_i\}) \times I \to (cp^n - \{p_i\})$$

where cp^n is a complex projective space, $\{p_i\}$ is any point belonging to cp^n, $I = [0,1]$. The retraction of the complex projective space cp^n is given by R: $(cp^n - \{p_i\}) \to A$, where A is given by

$$A = \left(0, 0, \cos\psi_2 \prod_{k=3}^{2n+1} \sin\psi_k, \ldots, \cos\psi_{i-1} \prod_{k=i}^{2n+1} \sin\psi_k,\right.$$

$$, \ldots, \cos\psi_{2n+2};\ -0,\ -0,\ -\cos\psi_2 \prod_{k=3}^{2n+1} \sin\psi_k$$

$$\left., \ldots,\ -\cos\psi_{i-1} \prod_{k=i}^{2n+1} \sin\psi_k, \ldots,\ -\cos\psi_{2n+2}\right)$$

$$= cp^{n-1} \quad (11)$$

we reach to the following theorem.

Theorem 1. The deformation retract of $\{cp^n - p_i\}$ onto a geodesic $cp^{n-1} \subset cp^n$ is

$$\Phi(m,t) = \left[\left(\cos\psi_1 \prod_{k=1}^{2n} \sin\psi_{k+1}, \prod_{k=1}^{2n+1} \sin\psi_k, \ldots,\right.\right.$$

$$\left.\cos\psi_{i-1} \prod_{k=i}^{2n+1} \sin\psi_k, \ldots, \cos\psi_{2n+1}\right); e^{i\pi}$$

$$\times \left(\cos\psi_1 \prod_{k=1}^{2n} \sin\psi_{k+1}, \prod_{k=1}^{2n+1} \sin\psi_k, \ldots,\right.$$

$$\left.\left.\cos\psi_{i-1} \prod_{k=i}^{2n+1} \sin\psi_k, \ldots, \cos\psi_{2n+1}\right)\right](1-t)$$

$$+ t\left(0, 0, \cos\psi_2 \prod_{k=3}^{2n+1} \sin\psi_k, \ldots, \cos\psi_{i-1}\right.$$

$$\times \prod_{k=1}^{2n+1} \sin\psi_k, \ldots, \cos\psi_{2n+1};\ -0,\ -0,$$

$$-\cos\psi_2 \prod_{k=3}^{2n+1} \sin\psi_k, \ldots,\ -\cos\psi_{i-1}$$

$$\left.\times \prod_{k=3}^{2n+1} \sin\psi_k, \ldots,\ -\cos\psi_{2n+1}\right) \quad (12)$$

where

$$\phi(m,0) = \{\beta, -\beta\}, \Phi(m,1) = \left(0, 0, \cos\psi_2 \prod_{k=3}^{2n+1}\right.$$

$$\times \sin\psi_k, \ldots, \cos\psi_{i-1} \prod_{k=i}^{2n+1} \sin\psi_k, \cos\psi_{2n+1};$$

$$0, 0,\ -\cos\psi_2 \prod_{k=3}^{2n+1} \sin\psi_k, \ldots,\ -\cos\psi_{i-1}$$

$$\left.\times \prod_{k=i}^{2n+1} \sin\psi_k,\ -\cos\psi_{2n+1}\right) \quad (13)$$

Now, we will discuss the folding of the complex projective space cp^n: $\mathscr{F}\colon cp^n \to cp^n$ where

$$\mathscr{F}(x_1, x_2, \ldots, x_{2n+2};\ -x_1,\ -x_2, \ldots,\ -x_{2n+2})$$

$$= (|x_1|, |x_2|, \ldots, x_{2n+2};\ -|x_1|,$$

$$-|x_2|, \ldots, , \ldots,\ -x_{2n+2}) \quad (14)$$

is an isometric folding of cp^n into itself, whence

$$\begin{aligned}\mathscr{F}:&\Bigg[\Bigg(\cos\psi_1\prod_{k=1}^{2n}\sin\psi_{k+1},\ \prod_{k=1}^{2n+1}\sin\psi_k,\ldots,\cos\psi_{i-1}\\&\times\prod_{k=i}^{2n+1}\sin\psi_k,\ldots\cos\psi_{2n+1};\ -\cos\psi_1\prod_{k=1}^{2n}\sin\psi_{k+1},\\&-\prod_{k=1}^{2n+1}\sin\psi_k,\ldots,\ -\cos\psi_{i-1}\prod_{k=i}^{2n+1}\sin\psi_k,\ldots,\\&-\cos\psi_{2n+1}\Bigg)-\{p_i\}\Bigg]\to\Bigg[\Bigg(\left|\cos\psi_1\prod_{k=1}^{2n}\sin\psi_{k+1}\right|,\\&\left|\prod_{k=1}^{2n+1}\sin\psi_k\right|,\ldots,\cos\psi_{i-1}\prod_{k=i}^{2n+1}\sin\psi_k,\ldots,\\&\cos\psi_{2n+1}\Bigg);\ -\left|\cos\psi_1\prod_{k=1}^{2n}\sin\psi_{k+1}\right|,\ -\left|\prod_{k=1}^{2n+1}\right.\\&\left.\times\sin\psi_k\right|,\ldots,\ -\cos\psi_{i-1}\prod_{k=i}^{2n+1}\sin\psi_k,\ldots,\\&-\cos\psi_{2n+1}\Bigg)-\{p_i\}\Bigg].\end{aligned}\tag{15}$$

The deformation retract of $\mathscr{F}(cp^n - p_i)$ will be defined as

$$\begin{aligned}\phi_{\mathscr{F}}:&\Bigg[\Bigg(\left|\cos\psi_1\prod_{k=1}^{2n}\sin\psi_{k+1}\right|,\left|\prod_{k=1}^{2n+1}\sin\psi_k\right|,\ldots,\cos\psi_{i-1}\\&\times\prod_{k=i}^{2n+1}\sin\psi_k,\ldots,\cos\psi_{2n+1};\ -\left|\cos\psi_1\right.\\&\left.\times\prod_{k=1}^{2n}\sin\psi_{k+1}\right|,\ -\left|\prod_{k=1}^{2n+1}\sin\psi_k\right|,\ldots,\ -\cos\psi_{i-1}\\&\times\prod_{k=1}^{2n+1}\sin\psi_k,\ldots,\ -\cos\psi_{2n+1}\Bigg)-\{p_i\}\Bigg]\times I\\&\to\Bigg[\Bigg(\left|\cos\psi_1\prod_{k=1}^{2n}\sin\psi_{k+1}\right|,\left|\prod_{k=1}^{2n+1}\sin\psi_k\right|\\&,\ldots,\cos\psi_{i-1}\prod_{k=i}^{2n+1}\sin\psi_k,\ldots,\cos\psi_{2n+1};\ -\left|\cos\psi_1\right.\\&\left.\times\prod_{k=1}^{2n}\sin\psi_{k+1}\right|,\ -\left|\prod_{k=1}^{2n+1}\sin\psi_k\right|,\ldots,\ -\cos\psi_{i-1}\\&\times\prod_{k=i}^{2n+1}\sin\psi_k,\ldots,\ -\cos\psi_{2n+1}\Bigg)-\{p_i\}\Bigg]\end{aligned}\tag{16}$$

with

$$\begin{aligned}\phi_{\mathscr{F}}(m,t)=&\Bigg[\Bigg(\left|\cos\psi_1\prod_{k=1}^{2n}\sin\psi_{k+1}\right|,\left|\prod_{k=1}^{2n+1}\sin\psi_k\right|\\&,\ldots,\cos\psi_{i-1}\prod_{k=i}^{2n+1}\sin\psi_k,\ldots,\cos\psi_{2n+1};\\&-\left|\cos\psi_1\prod_{k=1}^{2n}\sin\psi_{k+1}\right|,\ -\left|\prod_{k=1}^{2n+1}\sin\psi_k\right|\\&,\ldots,\ -\cos\psi_{i-1}\prod_{k=i}^{2n+1}\sin\psi_k,\ldots,\\&-\cos\psi_{2n+1}\Bigg)-\{p_i\}\Bigg](1-t)+t\Bigg[0,0,\\&\cos\psi_2\prod_{k=3}^{2n+1}\sin\psi_k,\ldots,\cos\psi_{i-1}\prod_{k=i}^{2n+1}\\&\times\sin\psi_k,\ldots,\cos\psi_{2n+1};\ -0,\ -0,\ -\cos\psi_2\\&\times\prod_{k=3}^{2n+1}\sin\psi_k,\ldots,\ -\cos\psi_{i-1}\\&\times\prod_{k=i}^{2n+1}\sin\psi_k,\ -\cos\psi_{2n+1}\Bigg]\end{aligned}\tag{17}$$

hence

$$\begin{aligned}\Phi_{\mathscr{F}}(m,1)=&\Bigg(0,0,\cos\psi_2\prod_{k=3}^{2n+2}\sin\psi_k,\ldots,\cos\psi_{i-1}\\&\times\prod_{k=i}^{2n+1}\sin\psi_k,\ldots,\cos\psi_{2n+1};\ -0,\ -0,\\&-\cos\psi_2\prod_{k=3}^{2n+1}\sin\psi_k,\ldots,\ -\cos\psi_{i-1}\\&\times\prod_{k=i}^{2n+1}\sin\psi_k,\ldots,\ -\cos\psi_{2n+1}\Bigg)\end{aligned}\tag{18}$$

which leads to Theorem 2.

Theorem 2. The folding of cp^n (Equation 14) and any folding homeomorphic to that folding, have the same deformation retract of the complex projective space cp^n onto a geodesic $cp^{n-1}\subset cp^n$.

Let the folding be defined as follows

$$\mathscr{F}^*\colon cp^n\to cp^n$$

$$\begin{aligned}&\mathscr{F}^*(x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_{2n+2};-x_1,\ -x_2,\ldots,\\&\quad -x_i,\ -x_{i+1},\ldots,\ -x_{2n+2})=(x_1,x_2,\ldots,|x_i|,\\&\quad |x_{i+1}|,\ldots,x_{2n+2};-x_1,\ -x_2,\ldots,\ -|x_i|,\\&\quad -|x_{i+1}|,\ldots,\ -x_{2n+2})\end{aligned}\tag{19}$$

An isometric folding of $cp^n-\{p_i\}$ into itself may be defined by

$$\begin{aligned}\mathscr{F}^*:&\Bigg[\Bigg(\cos\psi_1\prod_{k=1}^{2n}\sin\psi_{k+1},\ \prod_{k=1}^{2n+1}\sin\psi_k\\&,\ldots,\cos\psi_{i-1}\prod_{k=i}^{2n+1}\sin\psi_k,\cos\psi_i\prod_{k=i+1}^{2n+1}\sin\psi_k\\&,\ldots,\cos\psi_{2n+1};\ -\cos\psi_1\prod_{k=1}^{2n}\sin\psi_{k+1},\ldots,\\&-\cos\psi_{2n+1}\Bigg)-\{p_i\}\Bigg]\to\Bigg[\Bigg(\cos\psi_1\prod_{k=1}^{2n}\sin\psi_{k+1},\\&\prod_{k=1}^{2n+1}\sin\psi_k,\ldots,\left|\cos\psi_{i-1}\prod_{k=i}^{2n+1}\sin\psi_k\right|,\left|\cos\psi_i\right.\\&\left.\times\prod_{k=i+1}^{2n+1}\sin\psi_k\right|,\ldots,\cos\psi_{2n+1};-\cos\psi_1\prod_{k=1}^{2n}\\&\times\sin\psi_{k+1},\ldots,\ -\left|\cos\psi_{i-1}\prod_{k=i}^{2n+1}\sin\psi_k\right|,\ -\left|\cos\psi_i\right.\\&\left.\times\prod_{k=i+1}^{2n+1}\sin\psi_k\right|,\ldots,\ -\cos\psi_{2n+1}\Bigg)-\{p_i\}\Bigg]\end{aligned}\tag{20}$$

The deformation retract of this type of folding of

$cp^n - \{p_i\}$ will be defined by

$$\Phi_{\mathscr{F}^*}\colon \Bigg[\Bigg(\cos\psi_1 \prod_{k=1}^{2n} \sin\psi_{k+1}, \prod_{k=1}^{2n+1} \sin\psi_k, \ldots, \left|\cos\psi_{i-1} \prod_{k=i}^{2n+1} \sin\psi_k\right|, \left|\cos\psi_i \prod_{k=i+1}^{2n+1} \sin\psi_k\right|, \ldots, \cos\psi_{2n+1}; -\cos\psi_1 \prod_{k=1}^{2n} \sin\psi_{k+1}, \ldots, -\left|\cos\psi_{i-1} \prod_{k=i}^{2n+1} \sin\psi_k\right|, -\left|\cos\psi_i \prod_{k=i+1}^{2n+1} \sin\psi_k\right|, \ldots, -\cos\psi_{2n+1}\Bigg) - \{p_i\}\Bigg] \times I \rightarrow \Bigg[\Bigg(\cos\psi_1 \times \prod_{k=1}^{2n} \sin\psi_{k+1}, \prod_{k=1}^{2n+1} \sin\psi_k, \ldots, \left|\cos\psi_{i-1} \prod_{k=i}^{2n+1} \times \sin\psi_k\right|, \left|\cos\psi_i \prod_{k=i+1}^{2n+1} \sin\psi_k\right|, \ldots, \cos\psi_{2n+2}; -\cos\psi_1 \prod_{k=1}^{2n} \sin\psi_{k+1}, \ldots, -\cos\psi_{2n+2}\Bigg) - \{p_i\}\Bigg] \tag{21}$$

with

$$\Phi_{\mathscr{F}^*}(m,t) = \Bigg[\Bigg(\cos\psi_1 \prod_{k=1}^{2n} \sin\psi_{k+1}, \ldots, \left|\cos\psi_{i-1} \times \prod_{k=i}^{2n+1} \sin\psi_k\right|, \left|\cos\psi_i \prod_{k=i+1}^{2n+1} \sin\psi_k\right|, \ldots, \cos\psi_{2n+1}; -, -, -\cos\psi_{2n+1}\Bigg) - \{p_i\}\Bigg](1-t) + t\Bigg(0, 0, \cos\psi_2 \prod_{k=3}^{2n+1} \sin\psi_k, \ldots, \left|\cos\psi_{i-1} \prod_{k=i}^{2n+1} \sin\psi_k\right|, \left|\cos\psi_i \times \prod_{k=i+1}^{2n+1} \sin\psi_k\right|, \ldots, \cos\psi_{2n+1}; \ldots, -\cos\psi_{2n+1}\Bigg) \tag{22}$$

$$\Phi_{\mathscr{F}^*}(m,1) = \Bigg(0, 0, \cos\psi_2 \prod_{k=3}^{2n+1} \sin\psi_k, \ldots, \left|\cos\psi_{i-1} \times \prod_{k=i}^{2n+1} \sin\psi_k\right|, \left|\cos\psi_i \prod_{k=i+1}^{2n+1} \sin\psi_k\right|, \ldots, \cos\psi_{2n+1}; -0, -0, \ldots, -\left|\cos\psi_{i-1} \prod_{k=i}^{2n+1} \sin\psi_k\right|, -\left|\cos\psi_i \prod_{k=i+1}^{2n+1} \sin\psi_k\right|, \ldots, -\cos\psi_{2n+1}\Bigg) \tag{23}$$

We reach Theorem 3.

Theorem 3. The deformation retract of the isometric folding of cp^n defined in Equation 19 and any folding homeomorphic to this type of folding is different from the deformation retract of cp^n onto a geodesic $cp^{n-1} \subset p^n$.

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